

On the Fréchet derivative in elastic obstacle scattering

Frédérique Le Louër *

Abstract

In this paper, we investigate the existence and characterizations of the Fréchet derivatives of the solution to time-harmonic elastic scattering problems with respect to the boundary of the obstacle. Our analysis is based on a technique - the factorization of the difference of the far-field pattern for two different scatterers - introduced by Kress and Päivarinta to establish Fréchet differentiability in acoustic scattering. For the Dirichlet boundary condition an alternative proof of a differentiability result due to Charalambopoulos is provided and new results are proven for the Neumann and impedance exterior boundary value problems.

Keywords : Elastic scattering, Navier equation, Fréchet derivative, far-field pattern, Dirichlet condition, Neumann condition, impedance condition, inverse scattering.

1 Introduction

The inverse obstacle scattering problem for time harmonic waves is to determine the shape of the boundary and the location of a scatterer from far field measurements of the total wave. This problem is of practical interest in some important fields of applied physics, as for example non destructive testing in linear elasticity. Although such an inverse problem is theoretically difficult to solve since it is ill-posed and nonlinear, one can apply numerical methods to recover an approximate solution. The use of regularized iterative methods via first order linearization requires the Fréchet differentiability analysis of the far-field pattern of the solution to the forward problem with respect to the boundary of the scatterer. An explicit form of the first derivative is needed in view of its implementation in iterative algorithms.

In acoustic scattering, Fréchet differentiability with characterizations of the derivative as the far-field pattern of the radiating solution to a new exterior boundary value problem were investigated by Hettlich [10] and Kirsch [16] via variational methods, and by Hohage [12] via the implicit function theorem. These characterizations allow the numerical implementation of the derivatives from the knowledge of the boundary values of the scattered wave only (see [8, 12, 16]). By the use of boundary integral equation methods, one can express the far field pattern of the solution to scattering problems in terms of products

*Institut für Numerische und Angewandte Mathematik, Universität Göttingen, 37083 Göttingen, Germany, f.lelouer@math.uni-goettingen.de

of boundary integral operators with singular Schwarz kernels. The Fréchet differentiability analysis of the far field was then developed by Potthast [25, 27] for the Dirichlet and Neumann acoustic problems via the Fréchet differentiability analysis of the boundary integral operators involved, in the framework of Hölder continuous and differentiable function spaces. A characterization of the derivative can be obtained by directly deriving the boundary values of the solution. This approach was extended to electromagnetism by Potthast [26] for the perfect conductor problem and to elasticity by Charalambopoulos [2] for the Dirichlet scattering problem only. More recently, the Fréchet differentiability of the class of boundary integral operators with pseudohomogeneous hypersingular and weakly singular kernels - which includes the usual boundary integral operators occurring in time-harmonic potential theory - was analyzed by Costabel and Le Louër [4, 5, 22], in the framework of Sobolev spaces. The analyticity of the integral operators with respect to the boundary is proven. As a consequence, it yields the possibility to establish the Fréchet differentiability of the far-field pattern for any scattering problem by the use of boundary integral representations and the chain and product rules. In this way we obtain an additional implementable formula to compute the Fréchet derivatives of the far field by deriving the boundary integral operators (see [14, 15, 19]).

This paper is devoted to the Fréchet differentiability analysis of the far-field pattern of the solution to elastic obstacle scattering problems in three-dimensional homogeneous and isotropic media via an alternative technique introduced by Kress and Päivarinta in [18] to establish Fréchet differentiability for sound-soft and sound-hard obstacles. It is based on repeated uses of Green's theorem and a factorization of the difference of the far-field pattern of the scattered wave for a fixed obstacle and a perturbed obstacle. An interesting feature of the method is that it only requires the continuous dependence of the boundary values of the solution on the boundary in order to prove the Fréchet differentiability of the far-field pattern. This approach was extended to the perfect conductor problem by Kress in [17] and the impedance problem both in acoustic and electromagnetic scattering by Haddar and Kress in [7].

The paper is organized as follows : In section 2 we recall elementary results on time-harmonic Navier equations in Sobolev Spaces, following the notations of [1]. More details can be found in [6, 13]. The far field identity for elastic waves in the case of a Dirichlet boundary condition was established by Alves and Kress in [1]. In section 3, we use this identity to give an alternative proof of the differentiability result due to Charalambopoulos and we improve the boundary condition satisfied by the Fréchet derivative of the solution. In section 4 we apply this method - following ideas of Haddar and Kress - to establish the Fréchet differentiability of the boundary to far field operator simultaneously for the cases of Neumann and impedance boundary conditions and again we provide a characterization of the derivative.

2 The Navier equation

The propagation of time-harmonic elastic waves in the three-dimensional isotropic and homogeneous elastic medium characterized by the positive Lamé constants μ and λ and

the density ρ is described by the Navier equation

$$\operatorname{div} \sigma(\mathbf{u}) + \rho \omega^2 \mathbf{u} = 0, \quad (2.1)$$

where $\omega > 0$ is the frequency. Here

$$\sigma(\mathbf{u}) = \lambda(\operatorname{div} \mathbf{u})\mathbf{I}_3 + 2\mu\varepsilon(\mathbf{u}) \quad \text{and} \quad \varepsilon(\mathbf{u}) = \frac{1}{2}([\nabla \mathbf{u}] + [\nabla \mathbf{u}]^\top)$$

denote the stress tensor and the strain tensor respectively. Notice that \mathbf{I}_3 is the 3-by-3 identity matrix and $[\nabla \mathbf{u}]$ is the matrix whose the j -th column is the gradient of the j -th component of \mathbf{u} . We set $\Delta^* \mathbf{u} := \operatorname{div} \sigma(\mathbf{u}) = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u}$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary Γ of class \mathcal{C}^2 and outward unit normal vector \mathbf{n} and let Ω^c denote the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$. We denote by $H^s(\Omega)$, $H_{loc}^s(\overline{\Omega^c})$ and $H^s(\Gamma)$ the standard (local in the case of the exterior domain) complex valued, Hilbertian Sobolev space of order $s \in \mathbb{R}$ defined on Ω , $\overline{\Omega^c}$ and Γ respectively (with the convention $H^0 = L^2$.) Spaces of vector functions will be denoted by boldface letters, thus $\mathbf{H}^s = (H^s)^3$. We set :

$$\begin{aligned} \mathbf{H}(\Omega, \Delta^*) &:= \{ \mathbf{u} \in \mathbf{H}^1(\Omega) : \Delta^* \mathbf{u} \in \mathbf{L}^2(\Omega) \}, \\ \mathbf{H}_{loc}(\Omega^c, \Delta^*) &:= \{ \mathbf{u} \in \mathbf{H}_{loc}^1(\overline{\Omega^c}) : \Delta^* \mathbf{u} \in \mathbf{L}_{loc}^2(\overline{\Omega^c}) \}. \end{aligned}$$

The space $\mathbf{H}(\Omega, \Delta^*)$ is an Hilbert space endowed with the natural graph norm.

We use the following traces and tangential derivatives :

$$\begin{aligned} \frac{\partial}{\partial \mathbf{n}} &= \mathbf{n} \cdot \nabla \text{ (normal derivative),} \\ T(\mathbf{n}, \partial) &= 2\mu \frac{\partial}{\partial \mathbf{n}} + \lambda \mathbf{n} \operatorname{div} + \mu \mathbf{n} \times \mathbf{curl} \text{ (traction derivative),} \\ \mathcal{M}(\mathbf{n}, \partial) &= \frac{\partial}{\partial \mathbf{n}} - \mathbf{n} \operatorname{div} + \mathbf{n} \times \mathbf{curl} \text{ (tangential G\"unter's derivative).} \end{aligned}$$

The tangential gradient ∇_Γ and the surface divergence $\operatorname{div}_\Gamma$ are defined for a scalar function u and a vector function \mathbf{v} by the following equalities [24]:

$$\nabla u = \nabla_\Gamma u + \frac{\partial u}{\partial \mathbf{n}} \mathbf{n}, \quad \operatorname{div} \mathbf{v} = \operatorname{div}_\Gamma \mathbf{v} + \left(\mathbf{n} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \right),$$

and the tangential G\"unter's derivative can be rewritten as follow:

$$\mathcal{M} \mathbf{v} = [\nabla_\Gamma \mathbf{v}] \mathbf{n} - (\operatorname{div}_\Gamma \mathbf{v}) \mathbf{n}. \quad (2.2)$$

We note that, due to the trace lemma, $\mathbf{u}|_\Gamma \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ for $\mathbf{u} \in \mathbf{H}(\Omega, \Delta^*) \cup \mathbf{H}_{loc}(\Omega^c, \Delta^*)$. The normal derivative $\frac{\partial}{\partial \mathbf{n}} \mathbf{u}|_\Gamma$ and the traction derivative $T \mathbf{u}|_\Gamma$ are both defined as distributions in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ via the first Green formula (see [6, 21, 24] and lemma 2.1).

For two (3×3) matrices A and B whose columns are denoted by (a_1, a_2, a_3) and (b_1, b_2, b_3) , respectively, we set $A : B = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$. The following lemma is a consequence of the Gauss divergence theorem and the identity

$$\operatorname{div} (\sigma(\mathbf{u}) \cdot \mathbf{v}) = \Delta^* \mathbf{u} \cdot \mathbf{v} + \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}). \quad (2.3)$$

Lemma 2.1 For vector functions \mathbf{u} and \mathbf{v} in $\mathbf{H}(\Omega, \Delta^*)$, it holds the first Green formula

$$\int_{\Omega} \Delta^* \mathbf{u} \cdot \mathbf{v} \, dx = \int_{\Gamma} T \mathbf{u} \cdot \mathbf{v} \, ds - \int_{\Omega} \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx. \quad (2.4)$$

The symmetry of the product $\sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) = \lambda(\operatorname{div} \mathbf{u}) \cdot (\operatorname{div} \mathbf{v}) + 2\mu \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sigma(\mathbf{v}) : \varepsilon(\mathbf{u})$ yields the second Green formula

$$\int_{\Omega} (\mathbf{u} \cdot \Delta^* \mathbf{v} - \Delta^* \mathbf{u} \cdot \mathbf{v}) \, dx = \int_{\Gamma} (\mathbf{u} \cdot T \mathbf{v} - T \mathbf{u} \cdot \mathbf{v}) \, ds. \quad (2.5)$$

If \mathbf{u} and \mathbf{v} solve the Navier equation in Ω then each term in (2.5) vanish.

Further we will use the following different representations of the traction operator T :

$$T \mathbf{u} = \sigma(\mathbf{u}) \cdot \mathbf{n} = 2\mu \mathcal{M} \mathbf{u} + (\lambda + 2\mu)(\operatorname{div} \mathbf{u}) \mathbf{n} - \mu \mathbf{n} \times \operatorname{curl} \mathbf{u} \quad (2.6)$$

$$= \mu \left(\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \mathcal{M} \mathbf{u} \right) + (\lambda + \mu)(\operatorname{div} \mathbf{u}) \mathbf{n} \quad (2.7)$$

$$= (\lambda + 2\mu) \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - \lambda \mathcal{M} \mathbf{u} + (\lambda + \mu) \mathbf{n} \times \operatorname{curl} \mathbf{u}. \quad (2.8)$$

Now we assume that the domain Ω has a connected boundary Γ . In the sequel we are concerned with the following exterior boundary value problems for elastic waves: Given vector densities $f \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ and $g \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$, find a solution $\mathbf{u} \in \mathbf{H}_{loc}(\Omega^c, \Delta^*)$ to the Navier equation (2.1) in Ω^c which satisfies either a Dirichlet boundary condition

$$\mathbf{u} = f \quad \text{on } \Gamma \quad (2.9)$$

or an impedance boundary condition

$$T \mathbf{u} + i\alpha\omega\sqrt{\rho} \mathbf{u} = g \quad \text{on } \Gamma. \quad (2.10)$$

The impedance coefficient α is assumed to be a real non negative constant. The case $\alpha = 0$ yields the Neumann boundary condition. In addition the field \mathbf{u} has to satisfy the Kupradze radiation condition

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial \mathbf{u}_p}{\partial r} - i\kappa_p \mathbf{u}_p \right) = 0, \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial \mathbf{u}_s}{\partial r} - i\kappa_s \mathbf{u}_s \right) = 0, \quad r = |x|,$$

uniformly in all directions. Here, the longitudinal wave is given by $\mathbf{u}_p = -\kappa_p^{-2} \nabla \operatorname{div} \mathbf{u}$ and the transversal wave is given by $\mathbf{u}_s = \mathbf{u} - \mathbf{u}_p$ associated with the respective wave numbers κ_p and κ_s given by

$$\kappa_p^2 = \frac{\rho\omega^2}{\lambda + 2\mu}, \quad \kappa_s^2 = \frac{\rho\omega^2}{\mu}.$$

Solutions of the Navier equation satisfying the Kupradze radiation condition are called radiating solution.

The fundamental solution of the Navier equation is given by

$$\Phi(x, y) = \frac{1}{\mu} \left(\frac{e^{i\kappa_s|x-y|}}{4\pi|z|} \cdot \mathbf{I}_{\mathbb{R}^3} + \frac{1}{\kappa_s^2} \nabla_x \nabla_x^\top \left(\frac{e^{i\kappa_s|x-y|}}{4\pi|x-y|} - \frac{e^{i\kappa_p|x-y|}}{4\pi|x-y|} \right) \right).$$

It is a 3×3 matrix and for $j = 1, 2, 3$ we denote by Φ_j the j -th column of Φ . We have $\Phi(x, y) = \Phi(x, y)^\top = \Phi(y, x)$. From the second integral theorem (2.5), for a radiating solution $\mathbf{u} \in \mathbf{H}_{loc}(\Omega^c, \Delta^*)$ to the Navier equation (2.1), one can derive the Somigliana integral representation formula for $x \in \Omega^c$:

$$\mathbf{u}(x) = \int_{\Gamma} \left([T_y \Phi(x, y)]^\top \mathbf{u}(y) - \Phi(x, y) T_y \mathbf{u}(y) \right) ds(y), \quad (2.11)$$

where $T_y = T(\mathbf{n}(y), \partial_y)$ and $T_y \Phi(x, y)$ is the tensor obtained by applying the traction operator T_y to each column of $\Phi(x, y)$. For existence and uniqueness of a solution to the above boundary-value problems via boundary integral equation we refer to Kupradze [21]. The radiation condition implies that the solution has an asymptotic behavior of the form

$$\mathbf{u}(x) = \frac{e^{i\kappa_p|x|}}{|x|} \mathbf{u}_p^\infty(\hat{x}) + \frac{e^{i\kappa_s|x|}}{|x|} \mathbf{u}_s^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty,$$

uniformly in all directions $\hat{x} = \frac{x}{|x|}$. The fields \mathbf{u}_p^∞ and \mathbf{u}_s^∞ are defined on the unit sphere S^2 in \mathbb{R}^3 and known as the longitudinal and the transversal far-field pattern, respectively. We introduce the \mathbf{L}^2 -spaces

$$\begin{aligned} \mathbf{L}_s^2(S^2) &= \{\mathbf{h} \in \mathbf{L}^2(S^2); \mathbf{h}(\hat{x}) \cdot \hat{x} = 0\}, \\ \mathbf{L}_p^2(S^2) &= \{\mathbf{h} \in \mathbf{L}^2(S^2); \mathbf{h}(\hat{x}) \times \hat{x} = 0\}. \end{aligned}$$

We have $\mathbf{u}_s^\infty \in \mathbf{L}_s^2(S^2)$ and $\mathbf{u}_p^\infty \in \mathbf{L}_p^2(S^2)$.

3 The exterior Dirichlet boundary value problem

The scattering problem of time-harmonic waves by a bounded obstacle Ω leads to special cases of the above boundary value problems. In this section we consider the rigid body problem. The total displacement field $\mathbf{u} + \mathbf{u}^i$ is given by the superposition of the incident field \mathbf{u}^i , which we assume to be an entire solution of the Navier equation, and the scattered field \mathbf{u} , which solves the Navier equation in Ω^c , the Dirichlet boundary condition

$$\mathbf{u} + \mathbf{u}^i = 0 \quad \text{on } \Gamma,$$

and satisfies the Kupradze radiation condition.

For $x \in \Omega^c$ let W be the 3×3 matrix whose the j -th column W_j is the radiating solution of (2.1) and (2.9) for the boundary value

$$f = -\Phi_j(x, \cdot) \quad \text{on } \Gamma,$$

and set $V(x, \cdot) = \Phi(x, \cdot) + W(x, \cdot)$, that is W and V are the scattered and the total field, respectively, for the scattering of a point source located at $x \in \Omega^c$. We note that V satisfies the reciprocity

$$V(x, y) = [V(y, x)]^\top, \quad x, y \in \Omega^c, x \neq y,$$

which can be derived from the second Green formula (2.5), the Somigiana integral representation formula (4.2) and the symmetry of the fundamental solution.

Lemma 3.1 *The unique radiating solution $\mathbf{v} \in \mathbf{H}_{loc}(\Omega^c, \Delta^*)$ of the Navier equation (2.1) satisfying the boundary condition (2.9) for any $f \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ admits the following integral representation*

$$\mathbf{v}(x) = \int_{\Gamma} [T_y V(x, y)]^\top f(y) ds(y), \quad x \in \Omega^c. \quad (3.1)$$

PROOF. From the second Green formula (2.5) on Ω^c for the radiating solutions W and \mathbf{v} we can write

$$\int_{\Gamma} \left([T_y W(x, y)]^\top \mathbf{v}(y) - [W(x, y)]^\top T_y \mathbf{v}(y) \right) ds(y) = 0,$$

for all $x \in \Omega^c$. Using the boundary condition for W and \mathbf{v} , the symmetry of Φ and the Somigiana integral representation formula for \mathbf{v} we obtain

$$\begin{aligned} \mathbf{v}(x) &= \int_{\Gamma} \left([T_y \Phi(x, y)]^\top \mathbf{v}(y) + [W(x, y)]^\top T_y \mathbf{v}(y) \right) ds(y) \\ &= \int_{\Gamma} \left([T_y \Phi(x, y)]^\top \mathbf{v}(y) + [T_y W(x, y)]^\top \mathbf{v}(y) \right) ds(y) \\ &= \int_{\Gamma} [T_y V(x, y)]^\top \mathbf{v}(y) ds(y) = \int_{\Gamma} [T_y V(x, y)]^\top f(y) ds(y). \end{aligned}$$

■

For a fixed incident field \mathbf{u}^i , we consider the boundary to far field operator

$$F : \Gamma \mapsto \mathbf{u}^\infty = (\mathbf{u}_s^\infty, \mathbf{u}_p^\infty) \in \mathbf{L}_s^2(S^2) \times \mathbf{L}_p^2(S^2)$$

which maps the boundary of the rigid scatterer Ω onto the far-field patterns \mathbf{u}_s^∞ and \mathbf{u}_p^∞ of the scattered field \mathbf{u} . In order to describe the dependence of the operator F on the shape of the boundary Γ , we choose a fixed reference domain Ω and we consider variations generated by transformations of the form

$$x \mapsto x + \theta(x)$$

of point x in the space \mathbb{R}^3 , where θ is a smooth vector function defined in the neighborhood of Γ . The functions θ are assumed to be sufficiently small elements of the Banach space $\mathcal{C}^2(\Gamma, \mathbb{R}^3)$ in order that $(I + \theta)$ is a diffeomorphism from Γ to

$$\Gamma_\theta = (I + \theta)\Gamma = \{x + \theta(x); x \in \Gamma\},$$

so that the surface Γ_θ is a connected boundary of class \mathcal{C}^2 of a domain Ω_θ . The mapping $\mathcal{F} : \theta \mapsto F(\Gamma_\theta)$ is well defined in the neighborhood of the zero function in $\mathcal{C}^2(\Gamma, \mathbb{R}^3)$. We then analyze the Fréchet differentiability of \mathcal{F} at $\theta = 0$. We want to prove the existence of a linear and continuous mapping $\mathcal{F}'(0) : \mathcal{C}^2(\Gamma) \rightarrow \mathbf{L}_s^2(S^2) \times \mathbf{L}_p^2(S^2)$ such that we have the following expansion in $\mathbf{L}_s^2 \times \mathbf{L}_p^2$

$$\mathcal{F}(\xi) - \mathcal{F}(0) = \mathcal{F}'(0)\xi + o(\|\xi\|_{\mathcal{C}^2}), \quad \text{when } \|\xi\|_{\mathcal{C}^2} \rightarrow 0.$$

To this end, since we can interchange the differentiation with respect to the boundary and the passing to the limit $|x| \rightarrow \infty$, we will establish the Fréchet differentiability of the scattered field away from the boundary Γ .

By \mathbf{n}_θ we denote the exterior unit normal vector to Γ_θ and, in what follows, we will distinguish the quantities related to the exterior Dirichlet scattering problem for the domain Ω_θ through the subscript θ . We use the following identity established by Alves and Kress in [1] pp. 13.

Lemma 3.2 *Assume that $\overline{\Omega} \subset \Omega_\theta$. Then*

$$\mathbf{u}_\theta(x) - \mathbf{u}(x) = - \int_{\Gamma_\theta} [V(x, y)]^\top T_y(\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) ds(y), \quad (3.2)$$

for all $x \in \Omega_\theta^c$.

PROOF. From the second Green formula (2.5) for $\Phi(x, \cdot)$ and \mathbf{u}^i we can write

$$\int_{\Gamma} \left([T_y \Phi(x, y)]^\top \mathbf{u}^i(y) - \Phi(x, y) T_y \mathbf{u}^i(y) \right) ds(y) = 0,$$

for all $x \in \Omega^c$. Using the boundary condition of \mathbf{u} we then obtain

$$\mathbf{u}(x) = - \int_{\Gamma} \Phi(x, y) T_y(\mathbf{u}(y) + \mathbf{u}^i(y)) ds(y), \quad x \in \Omega^c. \quad (3.3)$$

From the second Green formula (2.5) for the radiating solutions W and \mathbf{u}_θ in Ω_θ^c we can write

$$\int_{\Gamma_\theta} \left([T_y W(x, y)]^\top \mathbf{u}_\theta(y) - [W(x, y)]^\top T_y \mathbf{u}_\theta(y) \right) ds(y) = 0, \quad x \in \Omega_\theta^c.$$

From the second Green formula (2.5) on $\Omega_\theta \setminus \overline{\Omega}$ for W and \mathbf{u}^i , the last equation and the boundary condition of \mathbf{u}_θ we have

$$\begin{aligned} & \int_{\Gamma_\theta} [W(x, y)]^\top T_y(\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) ds(y) \\ &= \int_{\Gamma_\theta} \left([W(x, y)]^\top T_y(\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) - [T_y W(x, y)]^\top (\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) \right) ds(y) \\ &= \int_{\Gamma_\theta} \left([W(x, y)]^\top T_y \mathbf{u}^i(y) - [T_y W(x, y)]^\top \mathbf{u}^i(y) \right) ds(y) \\ &= \int_{\Gamma} \left([W(x, y)]^\top T_y \mathbf{u}^i(y) - [T_y W(x, y)]^\top \mathbf{u}^i(y) \right) ds(y), \end{aligned} \quad (3.4)$$

for $x \in \Omega_\theta^c$. Using the boundary condition of W and \mathbf{u} on Γ , the integral representation (3.3) of \mathbf{u} and the second Green formula for radiating solutions we obtain

$$\begin{aligned}
-\mathbf{u}(x) &= - \int_{\Gamma} W(x, y) T_y (\mathbf{u}(y) + \mathbf{u}^i(y)) ds(y) \\
&= - \int_{\Gamma} \left([W(x, y)]^\top T_y \mathbf{u}^i(y) + [T_y W(x, y)]^\top \mathbf{u}(y) \right) ds(y) \\
&= - \int_{\Gamma} \left([W(x, y)]^\top T_y \mathbf{u}^i(y) - [T_y W(x, y)]^\top \mathbf{u}^i(y) \right) ds(y) \\
&= - \int_{\Gamma_\theta} [W(x, y)]^\top T_y (\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) ds(y),
\end{aligned}$$

for $x \in \Omega^c$. From (3.3) we can write for \mathbf{u}_θ

$$\mathbf{u}_\theta(x) = - \int_{\Gamma_\theta} \Phi(x, y) T_y (\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) ds(y), \quad x \in \Omega_\theta^c. \quad (3.5)$$

We obtain the identity (3.2) by combining the last two equations. ■

Remark 3.3 *As Kress and Päiväranta pointed out in [18] in the acoustic case, the lemma 3.2 remains valid when the domain Ω is not strictly contained in Ω_θ and W can be extended as a solution to the Navier equation in the exterior of Ω_θ . By theorem 5.7.1' in [23] pp. 169, this can be assured if Γ is analytic and Ω_θ does not differ too much from Ω . In this case the last equality in (4.4) follows by choosing an open domain D such that $\overline{\Omega} \cap \Omega_\theta \subset D$ and then applying Green's integral theorem first in $D \setminus \Omega$ and then in $D \setminus \Omega_\theta$.*

Lemma 3.4 *Assume that Γ is analytic. Then the following expansion holds*

$$\mathbf{u}_\theta - \mathbf{u} = \int_{\Gamma} [T_y V(\cdot, y)]^\top B \mathbf{u}(y) ds(y) + o(\|\theta\|_{\mathcal{C}^2}), \quad (3.6)$$

in $\mathbf{H}(G, \Delta^*)$ for all compact subset G of Ω^c and θ sufficiently small, where

$$B \mathbf{u} = -(\theta \cdot \mathbf{n}) \left(\frac{1}{\mu} (\mathbf{n} \times T(\mathbf{u} + \mathbf{u}^i)) \times \mathbf{n} + \frac{1}{\lambda + 2\mu} (\mathbf{n} \cdot T(\mathbf{u} + \mathbf{u}^i)) \mathbf{n} \right).$$

PROOF. We use similar arguments as in the proof of theorem 3.1 in [18] for the analogous acoustic case. We denote by S_θ and K'_θ the integral operators on the boundary Γ_θ with singular kernels $2\Phi(x, y)$ and $2[T_x \Phi(x, y)]$ respectively. The fundamental solution Φ is pseudo-homogeneous of class -1 . It can be shown that these operators are bounded from $\mathbf{H}^{-\frac{1}{2}}(\Gamma_\theta)$ to itself (see [6] and [24], pp. 176). From (3.5) and the jump relations, it can be deduced that the traction derivative of the total field $(\mathbf{u}_\theta + \mathbf{u}^i)$ solves the boundary integral equation

$$(I + K'_\theta - i\eta S_\theta) T(\mathbf{u}_\theta + \mathbf{u}^i_{|\Gamma_\theta}) = 2(T\mathbf{u}^i_{|\Gamma_\theta} - i\eta \mathbf{u}^i_{|\Gamma_\theta}).$$

The operator K'_θ is not compact, therefore we use a regularization technique (see [21]) to modify the integral equation as below

$$(Id + \mathcal{B}_\theta)T(\mathbf{u}_\theta + \mathbf{u}^i_{|\Gamma_\theta}) = 2\mathcal{H}_\theta(T\mathbf{u}^i_{|\Gamma_\theta} - i\eta\mathbf{u}^i_{|\Gamma_\theta})$$

where $(Id + \mathcal{B}_\theta) = \mathcal{H}_\theta(I + K'_\theta - i\eta S_\theta)$, \mathcal{H}_θ is a strongly integral operator and \mathcal{B}_θ is a weakly singular operator. The new integral operator $I + \mathcal{B}_\theta : \mathbf{H}^{-\frac{1}{2}}(\Gamma_\theta) \rightarrow \mathbf{H}^{-\frac{1}{2}}(\Gamma_\theta)$ is a Fredholm operator of the second kind which is invertible with bounded inverse. We use the transformation τ_θ which maps a function \mathbf{u}_θ defined on Γ_θ onto the function $\mathbf{u}_\theta \circ (I + \theta)$ defined on Γ . Reducing the analysis in [4] to the continuity and not the differentiability, we can prove that the boundary integral operators $\tau_\theta \mathcal{B}_\theta \tau_\theta^{-1}$ and $\tau_\theta \mathcal{H}_\theta \tau_\theta^{-1}$ depend continuously on the deformation $\theta \in \mathcal{C}^2(\Gamma)$ and so does the inverse $\tau_\theta (I + \mathcal{B}_\theta)^{-1} \tau_\theta^{-1}$ from the Neumann series. Since the incident field is analytic on the boundary Γ_θ , we then deduce that the total field satisfies

$$\|\tau_\theta (T(\mathbf{u}_\theta + \mathbf{u}^i)_{|\Gamma_\theta}) - T(\mathbf{u} + \mathbf{u}^i)\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \rightarrow 0, \quad \|\theta\|_{\mathcal{C}^2} \rightarrow 0 \quad (3.7)$$

Since Γ is analytic, by theorem 5.7.1' in [23] pp. 169 the total fields $(\mathbf{u} + \mathbf{u}^i)$ and W can be extended as a solution to the Navier equation across Γ in the exterior of $\overline{\Omega} \cap \Omega_\theta$. We also have

$$\|\tau_\theta (T(\mathbf{u} + \mathbf{u}^i)_{|\Gamma_\theta}) - T(\mathbf{u} + \mathbf{u}^i)\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \rightarrow 0, \quad \|\theta\|_{\mathcal{C}^2} \rightarrow 0. \quad (3.8)$$

By lemma 3.2 and using Taylor's formula together with (3.7)-(3.8), the boundary condition for V and the smoothness of $V(x, \cdot)$ up to Γ for x away from the boundary, it follows that

$$\int_{\Gamma_\theta} [V(\cdot, y)]^\top T_y(\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) ds(y) = \int_{\Gamma_\theta} [V(\cdot, y)]^\top T_y(\mathbf{u}(y) + \mathbf{u}^i(y)) ds(y) + o(\|\theta\|_{\mathcal{C}^2}),$$

in $\mathbf{H}(G, \Delta^*)$ for all compact subset G of Ω^c . From this, the first Green formula (2.4) and the boundary condition for W we obtain

$$\mathbf{u}_\theta - \mathbf{u} = - \int_{\Omega_\theta^*} \left\{ \left([\nabla V_j] : \sigma(\mathbf{u} + \mathbf{u}^i) \right)_{j=1,2,3} - \rho\omega^2 [V]^\top (\mathbf{u} + \mathbf{u}^i) \right\} \chi dy + o(\|\theta\|_{\mathcal{C}^2})$$

where

$$\Omega_\theta^* = \{y \in \Omega_\theta; y \notin \Omega\} \cup \{y \in \Omega; y \notin \Omega_\theta\},$$

and $\chi(y) = 1$ if $y \in \Omega_\theta$ and $y \notin \Omega$ and $\chi(y) = -1$ if $y \in \Omega$ and $y \notin \Omega_\theta$. Any $z \in \Omega_\theta^*$ can be represented of the form $z = y + t\theta(y)$, with $y \in \Gamma$ and $t > 0$. We have :

- (a) $\chi(z)dz = (\theta(y) \cdot \mathbf{n}(y)) ds(y) dt + o(\|\theta\|_{\mathcal{C}^2})$,
- (b) $(\mathbf{u} + \mathbf{u}^i)(y + t\theta(y)) = o(\|\theta\|_{\mathcal{C}^2})$ and $V(\cdot, y + t\theta(y)) = o(\|\theta\|_{\mathcal{C}^2})$,
- (c) $\nabla(\mathbf{u} + \mathbf{u}^i) = \frac{\partial}{\partial \mathbf{n}}(\mathbf{u} + \mathbf{u}^i) \cdot \mathbf{n}^\top + o(\|\theta\|_{\mathcal{C}^2})$ and,

(d) for $j = 1, 2, 3$,

$$[\nabla V_j] : \sigma(\mathbf{u} + \mathbf{u}^i) = \sigma(V_j) : [\nabla(\mathbf{u} + \mathbf{u}^i)] = T_y V_j \cdot \frac{\partial}{\partial \mathbf{n}}(\mathbf{u} + \mathbf{u}^i) + o(\|\theta\|_{\mathcal{C}^2}).$$

The expansion (a) for the volume form is a well-known result and we refer to [18, 7, 9] for a proof. Approximating the integral over Ω_θ^* by an integral over Γ we obtain that

$$\mathbf{u}_\theta - \mathbf{u} = - \int_\Gamma [T_y V(\cdot, y)]^\top \frac{\partial(\mathbf{u} + \mathbf{u}^i)}{\partial \mathbf{n}}(y) (\theta(y) \cdot \mathbf{n}(y)) ds(y) + o(\|\theta\|_{\mathcal{C}^2}).$$

To conclude, we express the normal derivative of $(\mathbf{u} + \mathbf{u}^i)$ in function of its traction derivative. Since $\mathbf{u} + \mathbf{u}^i = 0$ on Γ then the tangential Gunter's derivative $\mathcal{M}(\mathbf{u} + \mathbf{u}^i)|_\Gamma$ vanishes. Using (2.7), we obtain

$$(\mathbf{n} \times T(\mathbf{u} + \mathbf{u}^i)) \times \mathbf{n} = \mu \left(\mathbf{n} \times \frac{\partial}{\partial \mathbf{n}}(\mathbf{u} + \mathbf{u}^i) \right) \times \mathbf{n},$$

and using (2.8) we obtain

$$\mathbf{n} \cdot T(\mathbf{u} + \mathbf{u}^i) = (\lambda + 2\mu) \left(\mathbf{n} \cdot \frac{\partial(\mathbf{u} + \mathbf{u}^i)}{\partial \mathbf{n}} \right).$$

■

Theorem 3.5 *Let Γ be analytic. Then the mapping $\mathcal{F} : \mathcal{C}^2(\Gamma, \mathbb{R}^3) \rightarrow \mathbf{L}_s^2(S^2) \times \mathbf{L}_p^2(S^2)$ is Fréchet differentiable at $\theta = 0$ with the Fréchet derivative defined for $\xi \in \mathcal{C}^2(\Gamma, \mathbb{R}^3)$ by*

$$\mathcal{F}'(0)\xi = \mathbf{v}_\xi^\infty,$$

where \mathbf{v}_ξ^∞ is the far-field pattern of the solution \mathbf{v}_ξ to the Navier equation in Ω^c that satisfies the Kupradze radiation condition and the Dirichlet boundary condition

$$\mathbf{v}_\xi = -(\xi \cdot \mathbf{n}) \left(\frac{1}{\mu} (\mathbf{n} \times T(\mathbf{u} + \mathbf{u}^i)) \times \mathbf{n} + \frac{1}{\lambda + 2\mu} (\mathbf{n} \cdot T(\mathbf{u} + \mathbf{u}^i)) \mathbf{n} \right).$$

PROOF. This is a direct consequence of Lemmas 3.1 and 3.4. ■

Although theorem 3.5 was proven under the assumption that Γ is analytic, we expect that, proceeding as in [18], the result is also valid for \mathcal{C}^2 boundaries.

4 The exterior impedance boundary value problem

Now we consider the scattering problem by a cavity are an absorbing obstacle Ω . The total displacement field $\mathbf{u} + \mathbf{u}^i$ is given by the superposition of the incident field \mathbf{u}^i , which we assume to be an entire solution of the Navier equation, and the scattered field \mathbf{u} , which solves the Navier equation in Ω^c , the impedance boundary condition

$$T(\mathbf{u} + \mathbf{u}^i) + i\alpha\omega\sqrt{\rho}(\mathbf{u} + \mathbf{u}^i) = 0 \quad \text{on } \Gamma,$$

and satisfies the Kupradze radiation condition.

For $x \in \Omega^c$ let W be the 3×3 matrix whose the j -th column W_j is the radiating solution of (2.1) and (2.9) for the boundary value

$$g = -(T\Phi_j(x, \cdot) + i\alpha\omega\sqrt{\rho}\Phi_j(x, \cdot)) \quad \text{on } \Gamma,$$

and set $V(x, \cdot) = \Phi(x, \cdot) + W(x, \cdot)$.

Lemma 4.1 *The unique radiating solution $\mathbf{v} \in \mathbf{H}_{loc}(\Omega^c, \Delta^*)$ of the Navier equation (2.1) satisfying the boundary condition (2.10) for any $g \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ admits the following integral representation*

$$\mathbf{v}(x) = - \int_{\Gamma} [V(x, y)]^T g(y) ds(y), \quad x \in \Omega^c. \quad (4.1)$$

PROOF. From the second Green formula (2.5) on Ω^c for the radiating solutions W and \mathbf{v} we can write

$$\int_{\Gamma} \left([(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T \mathbf{v}(y) - [W(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{v}(y) \right) ds(y) = 0,$$

for all $x \in \Omega^c$. Using the boundary condition for W and \mathbf{v} and the Somigliana integral representation formula for \mathbf{v} we obtain

$$\begin{aligned} \mathbf{v}(x) &= \int_{\Gamma} \left([(T_y + i\alpha\omega\sqrt{\rho})\Phi(x, y)]^T \mathbf{v}(y) - [\Phi(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{v}(y) \right) ds(y) \\ &= \int_{\Gamma} \left(-[(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T \mathbf{v}(y) - [\Phi(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{v}(y) \right) ds(y) \\ &= - \int_{\Gamma} [V(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{v}(y) ds(y) = - \int_{\Gamma} [V(x, y)]^T g(y) ds(y). \end{aligned}$$

■

Here again we use a parametrization of the boundaries in order to investigate the Fréchet differentiability of the boundary to far field operator

$$F : \Gamma \mapsto \mathbf{u}^\infty = (\mathbf{u}_s^\infty, \mathbf{u}_p^\infty) \in \mathbf{L}_s^2(S^2) \times \mathbf{L}_p^2(S^2)$$

which maps the boundary of the Neumann or impedance obstacle Ω onto the far-field patterns \mathbf{u}_s^∞ and \mathbf{u}_p^∞ of the scattered field \mathbf{u} . Thus we will consider instead the mapping $\mathcal{F} : \theta \mapsto F(\Gamma_\theta)$. The following lemma give a factorization of the difference of the scattered field for two neighboring impedance obstacles.

Lemma 4.2 *Assume that $\overline{\Omega} \subset \Omega_\theta$. Then*

$$\mathbf{u}_\theta(x) - \mathbf{u}(x) = \int_{\Gamma_\theta} [T_y V(x, y) + i\alpha\omega\sqrt{\rho} V(x, y)]^T (\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) ds(y), \quad (4.2)$$

for all $x \in \Omega_\theta^c$.

PROOF. From the second Green formula (2.5) for $\Phi(x, \cdot)$ and \mathbf{u}^i we can write

$$\int_{\Gamma} \left([(T_y + i\alpha\omega\sqrt{\rho})\Phi(x, y)]^T \mathbf{u}^i(y) - \Phi(x, y)(T_y + i\alpha\omega\sqrt{\rho})\mathbf{u}^i(y) \right) ds(y) = 0,$$

for all $x \in \Omega^c$. Using the boundary condition of \mathbf{u} we then obtain

$$\mathbf{u}(x) = \int_{\Gamma} [(T_y + i\alpha\omega\sqrt{\rho})\Phi(x, y)]^T (\mathbf{u}(y) + \mathbf{u}^i(y)) ds(y), \quad x \in \Omega^c. \quad (4.3)$$

From the second Green formula (2.5) for the radiating solutions W and \mathbf{u}_{θ} in Ω_{θ}^c we can write for $x \in \Omega_{\theta}^c$

$$\int_{\Gamma_{\theta}} \left\{ [(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T \mathbf{u}_{\theta}(y) - [W(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{u}_{\theta}(y) \right\} ds(y) = 0.$$

From the second Green formula (2.5) for W and \mathbf{u}^i in $\Omega_{\theta} \setminus \overline{\Omega}$, the last equation and the boundary condition of \mathbf{u}_{θ} we have

$$\begin{aligned} & \int_{\Gamma_{\theta}} [(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T (\mathbf{u}_{\theta}(y) + \mathbf{u}^i(y)) ds(y) \\ &= \int_{\Gamma_{\theta}} \left([(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T (\mathbf{u}_{\theta}(y) + \mathbf{u}^i(y)) \right. \\ & \quad \left. - [W(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})(\mathbf{u}_{\theta}(y) + \mathbf{u}^i(y)) \right) ds(y) \\ &= \int_{\Gamma_{\theta}} \left([(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T \mathbf{u}^i(y) - [W(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{u}^i(y) \right) ds(y) \\ &= \int_{\Gamma} \left([(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T \mathbf{u}^i(y) - [W(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{u}^i(y) \right) ds(y), \end{aligned} \quad (4.4)$$

for $x \in \Omega_{\theta}^c$. Using the boundary condition of W and \mathbf{u} on Γ , the integral representation (4.3) of \mathbf{u} and the second Green formula for radiating solutions we obtain

$$\begin{aligned} -\mathbf{u}(x) &= \int_{\Gamma} [(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T (\mathbf{u}(y) + \mathbf{u}^i(y)) ds(y) \\ &= \int_{\Gamma} \left([(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T \mathbf{u}^i(y) + [W(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{u}(y) \right) ds(y) \\ &= \int_{\Gamma} \left([(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T \mathbf{u}^i(y) - [W(x, y)]^T (T_y + i\alpha\omega\sqrt{\rho})\mathbf{u}^i(y) \right) ds(y) \\ &= \int_{\Gamma_{\theta}} [(T_y + i\alpha\omega\sqrt{\rho})W(x, y)]^T (\mathbf{u}_{\theta}(y) + \mathbf{u}^i(y)) ds(y), \end{aligned}$$

for $x \in \Omega^c$. From (4.3) we can write for \mathbf{u}_{θ}

$$\mathbf{u}_{\theta}(x) = - \int_{\Gamma_{\theta}} [(T_y + i\alpha\omega\sqrt{\rho})\Phi(x, y)]^T (\mathbf{u}_{\theta}(y) + \mathbf{u}^i(y)) ds(y), \quad x \in \Omega_{\theta}^c. \quad (4.5)$$

We obtain the identity (4.2) by combining the last two equations. ■

The remark 3.3 is still available here, so that we expect that the identity (4.2) remains valid when the domain Ω is not strictly contained in Ω_θ .

We denote by H_Γ the mean curvature of Γ defined by

$$H_\Gamma = \operatorname{div}_\Gamma \mathbf{n}.$$

Lemma 4.3 *Assume that Γ is analytic. Then the following expansion holds*

$$\mathbf{u}_\theta(x) - \mathbf{u}(x) = - \int_\Gamma [V(x, y)] B\mathbf{u}(y) ds(y) + o(\|\theta\|_{\mathcal{C}^2(\Gamma, \mathbb{R}^3)}), \quad (4.6)$$

in $\mathbf{H}(G, \Delta^*)$ for all compact subset G of Ω^c and θ sufficiently small, where

$$\begin{aligned} B\mathbf{u} = & \operatorname{div}_\Gamma \left((\theta \cdot \mathbf{n}) \{ \sigma_1(\mathbf{u} + \mathbf{u}^i) - i\alpha\omega\sqrt{\rho}\mathbf{n} \cdot (\mathbf{u}_t + \mathbf{u}_t^i)^\top \} \right) \\ & + \rho\omega^2(\theta \cdot \mathbf{n}) \left(\left(1 - \frac{\alpha^2}{\mu}\right)(\mathbf{u}_t + \mathbf{u}_t^i) + \left(1 - \frac{\alpha^2}{\lambda + 2\mu}\right)(\mathbf{u}_n + \mathbf{u}_n^i)\mathbf{n} \right) \\ & + i\alpha\omega\sqrt{\rho}(\theta \cdot \mathbf{n}) \left([\nabla_\Gamma(\mathbf{u} + \mathbf{u}^i)]\mathbf{n} + \frac{\lambda}{\lambda + 2\mu}(\operatorname{div}_\Gamma(\mathbf{u} + \mathbf{u}^i))\mathbf{n} - (\mathbf{u} + \mathbf{u}^i)H_\Gamma \right), \end{aligned} \quad (4.7)$$

with

$$\sigma_1(\mathbf{u}) = \lambda \left(\frac{2\mu}{\lambda + 2\mu} \operatorname{div}_\Gamma \mathbf{u} - i \frac{\alpha\omega\sqrt{\rho}}{\lambda + 2\mu} \mathbf{u}_n \right) \Pi_3 + \mu \Pi_3 \left([\nabla_\Gamma \mathbf{u}] + [\nabla_\Gamma \mathbf{u}]^\top \right) \Pi_3, \quad (4.8)$$

and

$$\Pi_3 = \mathbf{I}_3 - \mathbf{n} \cdot \mathbf{n}^\top, \quad \mathbf{u}_t = (\mathbf{n} \times \mathbf{u}) \times \mathbf{n} \quad \text{and} \quad \mathbf{u}_n = \mathbf{u} \cdot \mathbf{n}.$$

PROOF. We follow the same procedure as in the proof of lemma 3.4. We denote by S_θ and K_θ the integral operators on the boundary Γ_θ with singular kernels $2\Phi(x, y)$ and $2[T_y \Phi(x, y)]^\top$ respectively. Assume that $\alpha > 0$. From (4.5) and the jump relations, it can be deduced that the restriction to Γ of the total field $(\mathbf{u}_\theta + \mathbf{u}^i)$ solves the boundary integral equation

$$(\mathbf{I} - K_\theta - i\alpha S_\theta)(\mathbf{u}_\theta + \mathbf{u}^i)|_{\Gamma_\theta} = 2\mathbf{u}^i|_{\Gamma_\theta}. \quad (4.9)$$

The operator K_θ is not compact. By regularization technique (see [21]), here again, we can modify the above equation in order to obtain an integral equation of the second kind which has to be solved for the unknown $\mathbf{u}_\theta + \mathbf{u}^i$ in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. From this new equation and the convergence of integral operators as $\theta \rightarrow 0$, it can be deduced that the total field satisfies

$$\|\tau_\theta(\mathbf{u}_\theta + \mathbf{u}^i)|_{\Gamma_\theta} - (\mathbf{u} + \mathbf{u}^i)|_\Gamma\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \rightarrow 0, \quad \|\theta\|_{\mathcal{C}^2} \rightarrow 0 \quad (4.10)$$

Then we can write that

$$\begin{aligned} & \int_{\Gamma_\theta} [(T_y + i\alpha\omega\sqrt{\rho})V(\cdot, y)]^\top (\mathbf{u}_\theta(y) + \mathbf{u}^i(y)) ds(y) \\ &= \int_{\Gamma_\theta} [(T_y + i\alpha\omega\sqrt{\rho})V(\cdot, y)]^\top (\mathbf{u}(y) + \mathbf{u}^i(y)) ds(y) + o(\|\theta\|_{\mathcal{C}^2}), \end{aligned}$$

in $\mathbf{H}(G, \Delta^*)$ for all compact subset G of Ω^c . Notice that the outer unit normal vector \mathbf{n} to Γ can be extended in a continuously differentiable function, denoted again by \mathbf{n} , on a tubular

$$B_T = \{z = y + t\mathbf{n}(y); y \in \Gamma, t \in [-T; T]\}$$

for some sufficiently small T . By the first Green formula (2.4) together with the following expansion for $y \in \Gamma$ (see [7, 9])

$$\mathbf{n}(y) \cdot (\mathbf{n}_\theta(y) - \mathbf{n}(y)) = -\mathbf{n}(y) \cdot \nabla_\Gamma(\theta(y) \cdot \mathbf{n}(y)) + o(\|\theta\|_{\mathcal{C}^2}) = o(\|\theta\|_{\mathcal{C}^2})$$

and the boundary condition for W we obtain

$$\mathbf{u}_\theta - \mathbf{u} = \int_{\Omega_\theta^*} \left\{ \operatorname{div} \left(\sigma(V_j)(\mathbf{u} + \mathbf{u}^i) + i\alpha\omega\sqrt{\rho}(V_j \cdot (\mathbf{u} + \mathbf{u}_i))\mathbf{n} \right)_{j=1,2,3} \right\} \chi dy + o(\|\theta\|_{\mathcal{C}^2})$$

where

$$\Omega_\theta^* = \{y \in \Omega_\theta; y \notin \Omega\} \cup \{y \in \Omega; y \notin \Omega_\theta\},$$

and $\chi(y) = 1$ if $y \in \Omega_\theta$ and $y \notin \Omega$ and $\chi(y) = -1$ if $y \in \Omega$ and $y \notin \Omega_\theta$. Approximating the integral over Ω_θ^* by an integral over Γ we obtain that

$$\begin{aligned} \mathbf{u}_\theta - \mathbf{u} &= \int_\Gamma \operatorname{div} \left(\sigma(V_j)(\mathbf{u} + \mathbf{u}^i) + i\alpha\omega\sqrt{\rho}(V_j \cdot (\mathbf{u} + \mathbf{u}_i))\mathbf{n} \right)_{j=1,2,3} (\theta \cdot \mathbf{n}) ds(y) + o(\|\theta\|_{\mathcal{C}^2}) \end{aligned}$$

For $j = 1, 2, 3$, we have

$$\begin{aligned} \operatorname{div} \left(\sigma(V_j)(\mathbf{u} + \mathbf{u}^i) \right) &= \sigma(V_j) : [\nabla(\mathbf{u} + \mathbf{u}^i)] - \rho\omega^2(V_j \cdot (\mathbf{u} + \mathbf{u}^i)) \\ &= [\nabla V_j] : \sigma(\mathbf{u} + \mathbf{u}^i) - \rho\omega^2(V_j \cdot (\mathbf{u} + \mathbf{u}^i)) \\ &= [\nabla_\Gamma(V_j)] : (\Pi_3 \sigma(\mathbf{u} + \mathbf{u}^i) \Pi_3) - \rho\omega^2(V_j \cdot (\mathbf{u} + \mathbf{u}^i)) \\ &\quad - i\alpha\omega\sqrt{\rho} \left(\frac{\partial}{\partial \mathbf{n}} V_j \cdot (\mathbf{u} + \mathbf{u}^i) + [\nabla_\Gamma V_j] : ((\mathbf{u}_t + \mathbf{u}_t^i) \cdot \mathbf{n}^\Gamma) \right) \end{aligned}$$

and

$$\operatorname{div} \left((V_j \cdot (\mathbf{u} + \mathbf{u}_i))\mathbf{n} \right) = \left(\frac{\partial}{\partial \mathbf{n}} + H_\Gamma \right) (V_j \cdot (\mathbf{u} + \mathbf{u}_i))$$

Collecting the two above equalities we obtain

$$\begin{aligned} \operatorname{div} \left(\sigma(V_j)(\mathbf{u} + \mathbf{u}^i) + i\alpha\omega\sqrt{\rho}(V_j \cdot (\mathbf{u} + \mathbf{u}_i))\mathbf{n} \right) &= [\nabla_\Gamma(V_j)] : (\Pi_3 \sigma(\mathbf{u} + \mathbf{u}^i) \Pi_3 - i\alpha\omega\sqrt{\rho}((\mathbf{u}_t + \mathbf{u}_t^i) \cdot \mathbf{n}^\Gamma)) \\ &\quad - \rho\omega^2(V_j \cdot (\mathbf{u} + \mathbf{u}^i)) + i\alpha\omega\sqrt{\rho}V_j \cdot \left(\frac{\partial}{\partial \mathbf{n}} + H_\Gamma \right) (\mathbf{u} + \mathbf{u}^i) \end{aligned} \quad (4.11)$$

To conclude we have to express $\Pi_3 \sigma(\mathbf{u} + \mathbf{u}^i) \Pi_3$ and $\frac{\partial}{\partial \mathbf{n}}(\mathbf{u} + \mathbf{u}^i)$ in function of the tangential derivatives of \mathbf{u} . First, we note that

$$\Pi_3 \sigma(\mathbf{u} + \mathbf{u}^i) \Pi_3 = \lambda \operatorname{div}(\mathbf{u} + \mathbf{u}^i) \Pi_3 + \mu \Pi_3 \left([\nabla_\Gamma \mathbf{u}] + [\nabla_\Gamma \mathbf{u}]^\Gamma \right) \Pi_3$$

Then we use the identity (2.6) together with the boundary condition of \mathbf{u} , which gives

$$\operatorname{div}(\mathbf{u} + \mathbf{u}^i) = \frac{2\mu}{\lambda + 2\mu} \operatorname{div}_\Gamma(\mathbf{u} + \mathbf{u}^i) - i \frac{\alpha}{\lambda + 2\mu} \omega \sqrt{\rho} \mathbf{n} \cdot (\mathbf{u} + \mathbf{u}^i).$$

The identity (2.7) yields

$$\left(\mathbf{n} \times \frac{\partial}{\partial \mathbf{n}}(\mathbf{u} + \mathbf{u}^i) \right) \times \mathbf{n} = -[\nabla_\Gamma(\mathbf{u} + \mathbf{u}^i)] \mathbf{n} - i \frac{\alpha}{\mu} \omega \sqrt{\rho} (\mathbf{n} \times (\mathbf{u} + \mathbf{u}^i)) \times \mathbf{n},$$

end the identity (2.8) yields

$$\mathbf{n} \cdot \frac{\partial}{\partial \mathbf{n}}(\mathbf{u} + \mathbf{u}^i) = -\frac{\lambda}{\lambda + 2\mu} \operatorname{div}_\Gamma(\mathbf{u} + \mathbf{u}^i) - i \frac{\alpha}{\lambda + 2\mu} \omega \sqrt{\rho} \mathbf{n} \cdot (\mathbf{u} + \mathbf{u}^i).$$

Substituting all the above identity in (4.11) we obtain the characterization (4.7). \blacksquare

Theorem 4.4 *Let Γ be analytic. Then the mapping $\mathcal{F} : \mathcal{C}^2(\Gamma, \mathbb{R}^3) \rightarrow \mathbf{L}_s^2(S^2) \times \mathbf{L}_p^2(S^2)$ is Fréchet differentiable at $\theta = 0$ with the Fréchet derivative defined for $\xi \in \mathcal{C}^2(\Gamma, \mathbb{R}^3)$ by*

$$\mathcal{F}'(0)\xi = \mathbf{v}_\xi^\infty,$$

where \mathbf{v}_ξ^∞ is the far-field pattern of the solution \mathbf{v}_ξ to the Navier equation in Ω^c that satisfies the Kupradze radiation condition and the impedance boundary condition

$$\begin{aligned} & (T + i\alpha\omega\sqrt{\rho})\mathbf{v}_\xi \\ &= \operatorname{div}_\Gamma \left((\xi \cdot \mathbf{n}) \{ \sigma_1(\mathbf{u} + \mathbf{u}^i) - i\alpha\omega\sqrt{\rho} \mathbf{n} \cdot (\mathbf{u}_t + \mathbf{u}_t^i)^\top \} \right) \\ &+ \rho\omega^2(\xi \cdot \mathbf{n}) \left(\left(1 - \frac{\alpha^2}{\mu}\right)(\mathbf{u}_t + \mathbf{u}_t^i) + \left(1 - \frac{\alpha^2}{\lambda + 2\mu}\right)(\mathbf{u}_n + \mathbf{u}_n^i)\mathbf{n} \right) \\ &+ i\alpha\omega\sqrt{\rho}(\xi \cdot \mathbf{n}) \left([\nabla_\Gamma(\mathbf{u} + \mathbf{u}^i)] \mathbf{n} + \frac{\lambda}{\lambda + 2\mu} (\operatorname{div}_\Gamma(\mathbf{u} + \mathbf{u}^i)) \mathbf{n} - (\mathbf{u} + \mathbf{u}^i) H_\Gamma \right), \end{aligned}$$

where the symmetric tensor $\sigma_1(\mathbf{u})$ is given by (4.8).

PROOF. This is a direct consequence of Lemmas 4.1 and 4.3. \blacksquare

When $\alpha = 0$, the integral equation (4.9) is not uniquely solvable but one can prove that the total field $(\mathbf{u}_\theta + \mathbf{u}^i)$ is the unique solution of an hypersingular boundary integral equation. Via regularization method one can prove that the estimation (4.10) is still valid in this case. For the Neumann problem, we then obtain the characterization:

Theorem 4.5 *Let Γ be analytic. Then the mapping $\mathcal{F} : \mathcal{C}^2(\Gamma, \mathbb{R}^3) \rightarrow \mathbf{L}_s^2(S^2) \times \mathbf{L}_p^2(S^2)$ is Fréchet differentiable at $\theta = 0$ with the Fréchet derivative defined for $\xi \in \mathcal{C}^2(\Gamma, \mathbb{R}^3)$ by*

$$\mathcal{F}'(0)\xi = \mathbf{v}_\xi^\infty,$$

where \mathbf{v}_ξ^∞ is the far-field pattern of the solution \mathbf{v}_ξ to the Navier equation in Ω^c that satisfies the Kupradze radiation condition and the Neumann boundary condition

$$T\mathbf{v}_\xi = \operatorname{div}_\Gamma ((\xi \cdot \mathbf{n}) \sigma_N(\mathbf{u} + \mathbf{u}^i)) + \rho\omega^2(\xi \cdot \mathbf{n})(\mathbf{u} + \mathbf{u}^i),$$

where

$$\sigma_N(\mathbf{u}) = \frac{2\lambda\mu}{\lambda + 2\mu} (\operatorname{div}_\Gamma \mathbf{u}) \Pi_3 + \mu \Pi_3 \left([\nabla_\Gamma \mathbf{u}] + [\nabla_\Gamma \mathbf{u}]^\top \right) \Pi_3. \quad (4.12)$$

5 Acknowledgements

This work was supported by the project SBF 755 "Nanoscale Photonic Imaging".

References

- [1] C. J. S. ALVES AND R. KRESS, *On the far-field operator in elastic obstacle scattering*, IMA J. Appl. Math., 67 (2002), pp. 1–21.
- [2] A. CHARALAMBOPOULOS, *On the Fréchet differentiability of boundary integral operators in the inverse elastic scattering problem*, Inverse Problems, 11 (1995), pp. 1137–1161.
- [3] D. COLTON AND R. KRESS, *Inverse acoustic and electromagnetic scattering theory*, vol. 93 of Applied Mathematical Sciences, Springer-Verlag, Berlin, second ed., 1998.
- [4] M. COSTABEL AND F. LE LOUËR, *Shape derivatives of boundary integral operators in electromagnetic scattering. Part I: Shape differentiability of pseudo-homogeneous boundary integral operators*, preprint, (2011). <http://tel.archives-ouvertes.fr/tel-00421863/fr/>.
- [5] ———, *Shape derivatives of boundary integral operators in electromagnetic scattering. Part II: Application to the scattering problem by a homogeneous dielectric obstacle*, preprint, (2011). <http://tel.archives-ouvertes.fr/tel-00421863/en/>.
- [6] M. COSTABEL AND E. P. STEPHAN, *Integral equations for transmission problems in linear elasticity*, J. Integral Equations Appl., 2 (1990), pp. 211–223.
- [7] H. HADDAR AND R. KRESS, *On the Fréchet derivative for obstacle scattering with an impedance boundary condition*, SIAM J. Appl. Math., 65 (2004), pp. 194–208 (electronic).
- [8] H. HARBRECHT AND T. HOHAGE, *Fast methods for three-dimensional inverse obstacle scattering problems*, J. Integral Equations Appl., 19 (2007), pp. 237–260.
- [9] A. HENROT AND M. PIERRE, *Variation et optimisation de formes*, vol. 48 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].
- [10] F. HETTLICH, *Fréchet derivatives in inverse obstacle scattering*, Inverse Problems, 11 (1995), pp. 371–382.
- [11] ———, *Erratum: Fréchet derivatives in inverse obstacle scattering*, Inverse Problems, 14 (1998), pp. 209–210.
- [12] T. HOHAGE, *Iterative Methods in Inverse Obstacle Scattering: Regularization Theory of Linear and Nonlinear Exponentially Ill-Posed Problems*, PhD in Numerical analysis, University of Linz, 1999.
- [13] G. C. HSIAO AND W. L. WENDLAND, *Boundary integral equations*, vol. 164 of Applied Mathematical Sciences, Springer-Verlag, Berlin, 2008.
- [14] O. IVANYSHYN AND T. JOHANSSON, *Nonlinear integral equation methods for the reconstruction of an acoustically sound-soft obstacle*, J. Integral Equations Appl., 19 (2007), pp. 289–308.
- [15] O. IVANYSHYN AND R. KRESS, *Identification of sound-soft 3D obstacles from phaseless data*, Inverse Probl. Imaging, 4 (2010), pp. 131–149.

- [16] A. KIRSCH, *The domain derivative and two applications in inverse scattering theory*, Inverse Problems, 9 (1993), pp. 81–96.
- [17] R. KRESS, *Electromagnetic waves scattering : Scattering by obstacles*, Scattering, (2001), pp. 191–210. Pike, E. R. and Sabatier, P. C., eds., Academic Press, London.
- [18] R. KRESS AND L. PÄIVÄRINTA, *On the far field in obstacle scattering*, SIAM J. Appl. Math., 59 (1999), pp. 1413–1426 (electronic).
- [19] R. KRESS AND W. RUNDELL, *Nonlinear integral equations and the iterative solution for an inverse boundary value problem*, Inverse Problems, 21 (2005), pp. 1207–1223.
- [20] V. D. KUPRADZE, *Potential methods in the theory of elasticity*, Translated from the Russian by H. Gutfreund. Translation edited by I. Meroz, Israel Program for Scientific Translations, Jerusalem, 1965.
- [21] V. D. KUPRADZE, T. G. GEGELIA, M. O. BASHELEISHVILI, AND T. V. BURCHULADZE, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*, vol. 25 of North-Holland Series in Applied Mathematics and Mechanics, North-Holland Publishing Co., Amsterdam, russian ed., 1979. Edited by V. D. Kupradze.
- [22] F. LE LOUËR, *Optimisation de formes d’antennes lentilles intégrées aux ondes millimétriques*, PhD in Numerical analysis, University of Rennes 1, 2009.
<http://tel.archives-ouvertes.fr/tel-00421863/fr/>.
- [23] C. B. MORREY, JR., *Multiple integrals in the calculus of variations*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1966 edition [MR0202511].
- [24] J.-C. NÉDÉLEC, *Acoustic and electromagnetic equations*, vol. 144 of Applied Mathematical Sciences, Springer-Verlag, New York, 2001. Integral representations for harmonic problems.
- [25] R. POTTHAST, *Fréchet differentiability of boundary integral operators in inverse acoustic scattering*, Inverse Problems, 10 (1994), pp. 431–447.
- [26] ———, *Domain derivatives in electromagnetic scattering*, Math. Methods Appl. Sci., 19 (1996), pp. 1157–1175.
- [27] ———, *Fréchet differentiability of the solution to the acoustic Neumann scattering problem with respect to the domain*, J. Inverse Ill-Posed Probl., 4 (1996), pp. 67–84.